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Relative Chebyshev Centers in Normed Linear Spaces, Part II*

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Let E be a normed linear space, A a bounded set in E, and G an arbitrary set in E. The relative Chebyshev center of A in G is the set of points in G best approximating A. We have obtained elsewhere general results characterizing the spaces in which the center reduces to a singleton in terms of structural properties related to uniform and strict convexity. In this paper, an analysis of the Chebyshev norm case, which falls outside the scope of the previous analysis, is presented.

INTRODUCTION

When E is a normed linear space and $A \subset E$ is bounded, the Chebyshev center of A is the set of elements of E best approximating A. When also $G \subset E$, we may consider the set of elements in G best approximating, from amongst all elements in G, the set A. This is called the relative Chebyshev center of A in G.

The first part of this work, [1], develops the connection among structural properties of relative centers, convexity properties of the spaces, and the

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closeness of the resemblance of the space to a pre-Hilbert space. This extends the work of Garkavi [7], of Day *et al.* [3] and of Rozema and Smith [17].

In the present paper we restrict our attention to the case where the space is $C[a, b]$ endowed with the uniform norm, i.e., we search for

$$\min_{u \in \mathcal{F}} \left(\max_{f \in S} \|f - u\|\right) = \min_{u \in \mathcal{F}} \max_{f \in S} \max_{t \in [a,b]} |f(t) - u(t)|,$$

where S is the set of functions to be approximated and $\mathcal{F}$ is the approximating family. This type of problem has been studied by several authors in recent years (e.g. [4, 6, 9, 10, 11, 13]). Mixed norms have also been discussed. For example, the problem of finding $\min(\|f_1 - u\|_\infty + \|f_2 - u\|_\infty)$, involving the l_1 and uniform norms, for two functions, has been investigated by Ling *et al.* [14]. Another, somewhat related problem, involves vectorial approximation (see, e.g., [2, 8]).

We focus our attention on the case where the approximating family is n-unisolvent. This is the natural framework for examining questions of uniqueness of best approximants.

We note that in spite of the fact that the general $C[a, b]$ problem can be reduced to a problem involving the approximation of two functions, essential differences exist between these problems and problems involving the approximation of one function, even where the range of the approximating functions is restricted (for an analysis of that problem, see [18]). These differences predicate a more complicated type of analysis, resulting in substantially different conditions for uniqueness, and a different type of characterization.

The approximating families will be taken as extended n-unisolvent (nonlinear) families, but the results are new even for the linear case of Tchebycheff systems. The proofs involve a somewhat delicate analysis of patterns of sign changes, and yield a full characterization of the situations where the center consists of exactly one element.

1. General Characterization of the Center

This section is devoted to a brief discussion of general results concerning centers in $C[0, 1]$. We recall the simple observation that in this particular norm a reduction to the case involving two functions, the upper and lower envelopes, is possible. We then present a proof of the characterization theorem for centers with respect to general n-unisolvent families, employing ideas to be utilized in the proof of the main theorem in Section 2.

The following simple observation has been made by several authors (see, e.g., [5]). When $A \subset C[0, 1]$ is compact, then the functions

$A_U(t) = \sup\{f(t); f \in A\}$ and $A_L(t) = \inf\{f(t); f \in A\}$ are continuous. Furthermore, when $g \in C[0, 1]$ we have

$$\begin{aligned} r(g, A) &= \sup\{\|f - g\|; f \in A\} \\ &= \sup\{|f(t) - g(t)|; f \in A, t \in [0, 1]\} \\ &= \sup\{\max(A_U(t) - g(t),\ g(t) - A_L(t); t \in [0, 1])\} \\ &= \max(\|A_U - g\|, \|g - A_L\|) = r(g; A_U, A_L). \end{aligned}$$

Hence, the problem of relative centers of compact sets in $C[0, 1]$ is reducible to a problem of relative centers for pairs of functions (f, g), with $f \geqslant g$. The latter type was discussed in a general framework in Section 2 of [1]. In the subsequent analysis we restrict ourselves to unisolvent n-parameter approximating families, and for the corresponding problems we establish existence, characterization and uniqueness properties.

Let $\mathcal{F} \subset C[0, 1]$ be an n-parameter approximating family, and define the relative center of (f, g) with respect to $\mathcal{F}$ (in the Chebyshev sense) by

$$Z(\mathcal{F}; f, g) = \{u^* \in \mathcal{F}; r(u^*; f, g) = \min(r(u; f, g); u \in \mathcal{F})\}. \quad (1.1)$$

Note that the existence of such u^* is assured by compactness; furthermore, it is assured even for families which are dense compact on X (i.e., families $\mathcal{G}$ such that every bounded sequence of elements of $\mathcal{G}$ has a subsequence converging pointwise on a dense subset Y of X to an element of $\mathcal{G}$). This was proved by Dunham [6].

We now restrict ourselves to unisolvent families. We start by recalling some of the relevant definitions and properties. For details and a thorough discussion of the place such families occupy in Approximation Theory, see, e.g., [15].

DEFINITION 1.1. The n-parameter approximating family $\mathcal{F} = \{F(\bar{a}; t); \bar{a} \in S \subset R^n\}$ of functions defined on $[0, 1]$ is *n-unisolvent* if for any given set $\{t_i\}_{i=1}^n$ of distinct points in $[0, 1]$ and any set $\{y_i\}_{i=1}^n$ of arbitrary numbers, there exists a unique $\bar{a}$ such that

$$F(\bar{a}; t_i) = y_i, \qquad i = 1, \ldots, n. \quad (1.2)$$

LEMMA 1.2 (see [15, p. 72]). *The solution $F(\bar{a}; t)$ of (1.2) is a continuous function of the t_i's and the y_i's; i.e., given $\varepsilon > 0$, $\bar{t}$, $\bar{y}$, there, exists a $\delta > 0$ such that*

$$\max(\|\bar{t} - \bar{t}'\|, \|\bar{y} - \bar{y}'\|) < \delta \Rightarrow \|F(\bar{a}; t) - F(\bar{a}'; t)\|_\infty < \varepsilon, \quad (1.3)$$

where $\bar{a}'$ is the solution of (1.2) for $\bar{t}'$, $\bar{y}'$.

Applying the standard limit argument used for T-systems, we deduce:

COROLLARY 1.3. *If $\mathcal{F}$ is n-unisolvent and $\bar{a} \neq \bar{b}$, then $F(\bar{a}; t) - F(\bar{b}; t)$ has at most $n-1$ zeros in $[0, 1]$. Here non-nodal zeros are counted twice (an interior point t_0 is a non-nodal zero of f if $f(t_0) = 0$ and f does not change sign at t_0).*

We conclude that for a fixed $\bar{t}$, the mapping $\bar{y} \to F(\bar{a}; \cdot)$ is a homeomorphism of S onto $\mathcal{F}$. Hence, each compact set in $C[0, 1]$ has a relative Chebyshev center in $\mathcal{F}$. We recall, furthermore, that analogues of the classical results for Chebyshev sets are valid for general n-unisolvent families, to-wit,

LEMMA 1.4 [15, p. 93]. *Let $\mathcal{F}$ be n-unisolvent on $[0, 1]$ and let $f \in C[0, 1]$. Then f possesses a unique best Chebyshev approximation characterized by the existence of an $n+1$-point alternance.*

Coming back to the problem at hand, we introduce now some additional notations and definitions, tailored for our needs.

DEFINITION 1.5. The set $(t_1, \ldots, t_k)$, $t_1 < t_2 < \cdots < t_k$, is called a *k-point alternance* for the approximation by u to f and g (abbreviated as the $(u; f, g)$-approximation) if either

$$r(u; f, g) = f(t_1) - u(t_1) = u(t_2) - g(t_2) = f(t_3) - u(t_3) = \cdots$$

or

$$r(u; f, g) = u(t_1) - g(t_1) = f(t_2) - u(t_2) = u(t_3) - g(t_3) = \cdots . \tag{1.4}$$

A point t_0 such that $f(t_0) - u(t_0) = r(u; f, g)$ is called a *(+)-point*, while a point t_0 such that $u(t_0) - g(t_0) = r(u; f, g)$ is called a *(−)-point*. Both kinds are called *(e)-points*. Following Dunham [5], we introduce the following definition:

DEFINITION 1.6. The point t_0 is called a *straddle* point with respect to the $(u; f, g)$-approximation if it is both a (+)- and a (−)-point, i.e., if

$$f(t_0) - u(t_0) = u(t_0) - g(t_0) = r(u; f, g). \tag{1.5}$$

We are now ready to state the first theorem for relative centers of (f, g) with respect to unisolvent families. The theorem is due to Dunham [5]. We present here our own proof, which is different from Dunham's, since our methods will be used subsequently to obtain further results.

THEOREM 1.7 [5]. *Let $f, g \in C[0, 1]$, with $f \geqslant g$, and let $\mathcal{F}$ be an n-unisolvent family in $C[0, 1]$. Then $u^* \in Z(\mathcal{F}; f, g)$ if, and only if, either:*

(a) $(u^*; f, g)$ *has a straddle point, or* (b) $(u^*; f, g)$ *has an* $n+1$*-point alternance. In the latter case,* $Z(\mathcal{F}; f, g) = \{u^*\}$.

Proof. Sufficiency: If $(u^*; f, g)$ has the straddle point t_0, then for each u

$$r(u; f, g) \geqslant \tfrac{1}{2}(f-g)(t_0) = r(u^*; f, g),$$

completing the proof in this case. Suppose next that $(u^*; f, g)$ has the $n+1$-point alternance $t_0 \cdots t_n$, and assume for definiteness that

$$f(t_0) - u^*(t_0) = u^*(t_1) - g(t_1) = \cdots = r(u^*; f, g). \tag{1.6}$$

For each $u \in Z(\mathcal{F}; f, g)$ we must have

$$\max[f(t_i) - u(t_i), u(t_i) - g(t_i)] \leqslant r(u^*; f, g), \qquad i = 0, 1, \ldots, n.$$

Combining with (1.6), this yields

$$(-1)^{i+1}[u(t_i) - u^*(t_i)] \geqslant 0, \qquad i = 0, 1, \ldots, n.$$

Thus, $u^* - u$ has at least n zeros (where multiplicities are counted as in Corollary 1.3). This is possible, in view of Corollary 1.3, only if $u^* = u$. Hence, u^* is the only element of the center.

Necessity: Assume that u^* has no straddle points and that it has only $k+1$ points of alternance, $0 \leqslant k < n$. Since $(u^*; f, g)$ has no straddle points, we have

$$2r(u^*; f, g) - \|f - g\| = 5\delta > 0. \tag{1.7}$$

With no loss of generality we may assume that the first (e)-point t_0 is a $(+)$-point. We then sequentially define

$$t_0 = \min\{t; t \text{ is a } (+)\text{-point}\},$$

$$\underline{t}_0 = \min\{t; f(t) - u^*(t) \geqslant r(u^*; f, g) - 2\delta\},$$

$$t_1 = \min\{t; t \text{ is a } (-)\text{-point}\},$$

$$\bar{t}_0 = \max\{t; t < t_1, f(t) - u^*(t) \geqslant r(u^*; f, g) - 2\delta\},$$

$$\underline{t}_1 = \min\{t; t > \bar{t}_0, u^*(t) - g(t) \geqslant r(u^*; f, g) - 2\delta\},$$

$$t_2 = \min\{t; t > t_1, t \text{ is a } (+)\text{-point}\},$$

etc. Now let $A_i = [\underline{t}_i, \bar{t}_i]$, $i = 0, \ldots, k$. There are $k+1$ such intervals since each interval contains precisely one of the alternance points. Observe that the A_i's are disjoint closed intervals satisfying $A_0 < A_1 < \cdots < A_k$.

Furthermore, all (+)-points are in $\bigcup A_{2i}$, while all (−)-points are in $\bigcup A_{2i+1}$. Note finally that

$$\max(|f(t)-u^*(t)|, |u^*(t)-g(t)|) < r(u^*; f, g) - 2\delta, \quad \text{for all } t \in [0,1] \setminus \bigcup_{i=0}^{k} A_i. \tag{1.8}$$

Now choose a sequence of points $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_{n-1}$ satisfying the conditions

(a) $\tilde{t}_i \in (\underline{t}_{i-1}, \underline{t}_i)$, $1 \leqslant i \leqslant k$,

(b) if $n \not\equiv k \pmod 2$, then

$$\tilde{t}_i \in (\underline{t}_{k-1}, \underline{t}_k), \qquad k+1 \leqslant i \leqslant n-1. \tag{1.9}$$

If $n \equiv k \pmod 2$, then (1.9) is required to hold for $i \leqslant n-2$, and $\tilde{t}_{n-1} = 1$.

We discuss first the case $n \not\equiv k$. We adjoin a point $\tilde{t}_0$ in A_0, and construct a function $u \in \mathcal{F}$ satisfying

$$u(\tilde{t}_0) = u^*(\tilde{t}_0) + \eta,$$
$$u(\tilde{t}_i) = u^*(\tilde{t}_i), \qquad i = 1,\ldots, n-1,$$

where η is chosen so small that $\|u - u^*\| < \delta$. This is possible in view of the continuity properties expressed in Lemma 1.2.

Note that $u - u^*$ cannot vanish at any point other than the $\tilde{t}_i$'s in view of the unisolvence. Hence, $u(t) > u^*(t)$ on A_0, and in view of the way the $\tilde{t}_i$'s are placed, we have

$$(-1)^i[u(t) - u^*(t)] > 0 \qquad \text{if} \quad t \in A_i,\ i = 0, 1,\ldots, k.$$

Thus, we obtain

$$\begin{cases} 0 < r(u^*; f, g) - \delta < f(t) - u(t) < f(t) - u^*(t), & t \in A_{2i}, \\ & i = 0, 1,\ldots, k-1. \\ 0 < r(u^*; f, g) - \delta < u(t) - g(t) < u^*(t) - g(t), & t \in A_{2i+1}, \end{cases}$$

In the complement of $\bigcup_{i=0}^{k} A_i$, we clearly have

$$\max(|f-u|, |u-g|) < r(u^*; f, g).$$

Combining these inequalities, we conclude that $r(u; f, g) < r(u^*; f, g)$, i.e., that u^* is not in the center.

The second case is similarly handled. Q.E.D.

COROLLARY 1.8. *The point $\tilde{t}$ is a straddle point of some triplet $(u; f, g)$, if and only if*

$$f(\tilde{t}) - g(\tilde{t}) = 2r(\mathcal{F}; f, g). \tag{1.10}$$

Thus, if $\tilde{t}$ is a straddle point for one triplet, it is a straddle point for all triplets, and $u^*(\tilde{t}) = [f(\tilde{t}) + g(\tilde{t})]/2$ for all $u^* \in Z(\mathcal{F}; f, g)$.

Proof. Suppose that (1.10) is satisfied, and let $u^* \in Z(\mathcal{F}; f, g)$. Then

$$\max[f(\tilde{t}) - u^*(\tilde{t}), u^*(\tilde{t}) - g(\tilde{t})] \leqslant r(u^*; f, g) = r(\mathcal{F}; f, g).$$

Combining this with (1.10), it follows that

$$f(\tilde{t}) - u(\tilde{t}) = u(\tilde{t}) - g(\tilde{t}) = r(u^*; f, g) = r(\mathcal{F}; f, g), \tag{1.11}$$

so that $\tilde{t}$ is a straddle point of $(u^*; f, g)$. Conversely, if $\tilde{t}$ is a straddle point of $(u^*; f, g)$, then by the previous theorem, $u^* \in Z(\mathcal{F}; f, g)$, and using (1.11) we have (1.10).

The last observation in the corollary is a consequence of (1.11). Q.E.D.

2. UNIQUENESS

We examine in this section the conditions under which the center will reduce to a singleton. It will be shown that the situation here is more complicated than the corresponding one in the approximation of one function, and an analogue does not exist. An intermediate situation, where some of the difficulties are beginning to show, occurs in the study of the approximation of discontinuous functions (see, e.g., [16]).

The first result we have in this direction is a simple consequence of the definition of n-unisolvence and Corollary 1.8.

LEMMA 2.1. *Let $\mathcal{F}$ be an n-unisolvent family and let $f, g, f \geqslant g$, be two continuous functions. If there exist n straddle points (i.e., points satisfying (1.10)) then $Z(\mathcal{F}; f, g)$ is a singleton.*

The complete analysis of the conditions under which $Z(\mathcal{F}; f, g)$ is a singleton requires more than standard perturbation methods, due to special phenomena which do not have a counterpart in the theory of approximation of one function. Even the theory of approximation of one function by functions with restricted ranges (see, e.g., Sippel [18]) does not exhibit these difficulties, and has substantially different uniqueness characteristics. As a simple example of the special phenomena we have here, we observe that the subsequent discussion implies that if f, g are continuously differentiable and

$\mathcal{F} = [1, x]$, then the existence of *one* interior straddle point suffices to ensure that $Z(\mathcal{F}; f, g)$ is a singleton.

We consider first the simplest case, where all the functions under consideration are n-times differentiable.

DEFINITION 2.2. The n-parameter family $\mathcal{F}$ of n-times differentiable functions will be called an *extended n-unisolvent family* if for any prescribed set of "Hermite-data," i.e., data of the form

$$u^{(j)}(t_i) = \alpha_i^j, \qquad i = 1,..., m; j = 0, 1,..., k_i - 1; \sum_{i=1}^{m} k_i = n \tag{2.1}$$

there exists a unique $u \in \mathcal{F}$ satisfying (2.1).

This generalizes, to unisolvent families, the concept of an Extended Tchebycheff system, which proved useful in the study of Tchebycheff systems (see [12]). Naturally, each Extended Tchebycheff system is an extended n-unisolvent family.

Remark. The natural analogue of Lemma 1.2 is valid for extended n-unisolvent families.

DEFINITION 2.3. Let $\mathcal{F}$ be an extended n-unisolvent family, and let $f \geqslant g$, where $f, g \in C^n(I)$. The straddle point t_0 has the *deficiency index k*, $k \leqslant n$, if k is the largest integer such that

$$u \in Z(\mathcal{F}; f, g) \Rightarrow u^{(j)}(t_0) = f^{(j)}(t_0) = g^{(j)}(t_0), \qquad j = 1,..., k-1. \tag{2.2}$$

Since $u \in Z(\mathcal{F}; f, g)$ implies that $u(t_0) = ((f + g)(t_0))/2$ at a straddle point t_0 of deficiency k, u has to satisfy k Hermite-type conditions there.

LEMMA 2.4. *Let $\mathcal{F}$ be an extended n-unisolvent family, and let $f \geqslant g$, where $f, g \in C^n(I)$. Let t_0 be an interior straddle point, with deficiency index k. Then $2 \leqslant k \leqslant n$. If $k < n$, then k is even.*

Proof. Let $u^* \in Z(\mathcal{F}; f, g)$. Then

$$f'(t_0) = u^{*\prime}(t_0) = g'(t_0). \tag{2.3}$$

Indeed, assuming that $f'(t_0) > u^{*\prime}(t_0)$, we observe that in a small right neighborhood of t_0, the inequality $f(t) - u^*(t) > f(t_0) - u^*(t_0)$ is valid in contradiction to the assumption that

$$f(t_0) - u^*(t_0) = r(u^*; f, g).$$

A similar analysis, involving the left neighborhood, is obtained if $f'(t_0) < u^{*\prime}(t_0)$. Hence, $f'(t_0) = u^{*\prime}(t_0)$. The right-hand side equality is similarly derived. Note that this type of result does not extend to second order derivatives, where only the weak inequalities $f''(t_0) \leqslant u^{*\prime\prime}(t_0) \leqslant g''(t_0)$

have to hold. If $f''(t_0) = g''(t_0)$, then the chain collapses, and $u^{*\prime\prime}(t_0)$ has to take the common value. If

$$f^{(2j)}(t_0) = u^{(2j)}(t_0) = g^{(2j)}(t_0) \qquad \text{and} \qquad 2j < n-1,$$

then the argument used in proving (2.3) now yields

$$f^{(2j+1)}(t_0) = u^{(2j+1)}(t_0) = g^{(2j+1)}(t_0)$$

establishing the evenness of k. Q.E.D.

Remarks. 1. If the straddle point t_0 is an endpoint, then the deficiency index k does not have to be greater than 1. Similar arguments applied to the kth derivative at t_0 yield only

$$f^{(k)}(t_0) \leqslant u^{(k)}(t_0) \leqslant g^{(k)}(t_0). \tag{2.4}$$

2. The concept of a deficiency index can be extended to the case when f and g are not smooth. It is defined then as the number of Hermite-type conditions that have to be satisfied by the elements of $Z(\mathcal{F}; f, g)$ at t_0. Lemma 2.4 is no longer valid, since the deficiency index may depend on the degree of smoothness. The analysis in this case proceeds along similar lines and involves one-sided Dini derivatives.

We proceed to define the concepts of a boundary straddle point, deficiency induced by an element of the center, and total deficiency. The need for these stems from the observation that a straddle point may be a cluster point of (e)-points, limiting the freedom to perturb functions of the center.

LEMMA 2.5. *Let $\tilde{u} \in Z(\mathcal{F}; f, g)$ and let y be a straddle point of deficiency k. If*

$$\tilde{u}^{(j)}(y) = g^{(j)}(y), \qquad j = k, k+1, \ldots, k+m-1, \; m \geqslant 1, \tag{2.5}$$

and y is a cluster point of $(+)$-points, then

$$u^{(j)}(y) = g^{(j)}(y), \qquad j = k, k+1, \ldots, k+m-1, \tag{2.6}$$

for all $u \in Z(\mathcal{F}; f, g)$. Similarly, if

$$\tilde{u}^{(j)}(y) = f^{(j)}(y), \qquad j = k, k+1, \ldots, k+m-1, \tag{2.5$'$}$$

and y is a cluster point of $(-)$-points, then

$$u^{(j)}(y) = f^{(j)}(y), \qquad j = k, k+1, \ldots, k+m-1, \tag{2.6$'$}$$

for all $u \in Z(\mathcal{F}; f, g)$.

Proof. We prove (2.6), the proof for the other case being similar. The proof proceeds by induction on m. We start with $m = 1$, and note that the

fact that y has deficiency k and relation (2.4) imply that for any $u \in Z(\mathcal{F}; f, g)$

$$\text{and} \qquad \begin{aligned} u^{(j)}(y) &= \tilde{u}^{(j)}(y), \qquad j = 0, 1, \ldots, k-1, \\ u^{(k)}(y) &\leqslant g^{(k)}(y) = \tilde{u}^{(k)}(y). \end{aligned} \tag{2.7}$$

If $u^{(k)}(y) < \tilde{u}^{(k)}(y)$ then, in view of (2.7), in a sufficiently small neighborhood of y, $u(t) < \tilde{u}(t)$. Now let t^* be a $(+)$-point of $\tilde{u}$ lying in this neighborhood. Then we have the chain of inequalities

$$r(u; f, g) \geqslant \|f - u\| \geqslant (f - u)(t^*) > (f - \tilde{u})(t^*) = \|f - \tilde{u}\| = r(\mathcal{F}; f, g),$$

contradicting the assumption that $\tilde{u} \in Z(\mathcal{F}; f, g)$. Hence (2.6) must hold for $m = 1$. The induction step is similar, proving (2.6) for general m. Q.E.D.

We are thus led to the following definition.

DEFINITION 2.6. (a) Let $\tilde{u} \in Z(\mathcal{F}; f, g)$ and let y be a straddle point of deficiency k, which is a cluster point of (e)-points. If y is a cluster point of $(+)$-points and m is the largest integer $0 \leqslant m \leqslant n - k$ such that

$$\tilde{u}^{(j)}(y) = g^{(j)}(y), \qquad j = 0, 1, \ldots, k + m - 1, \tag{2.8}$$

then m is called the *upper $\tilde{u}$-induced deficiency* of y. If y is a cluster point of $(-)$-points and m is the largest integer $0 \leqslant m \leqslant n - k$ such that

$$\tilde{u}^{(j)}(y) = f^{(j)}(y), \qquad j = 0, k, \ldots, k + m - 1, \tag{2.9}$$

then m is called the *lower $\tilde{u}$-induced deficiency* of y.

(b) Let $\tilde{u} \in Z(\mathcal{F}; f, g)$ and let y be a straddle point of deficiency k which is not a cluster point of $(+)$-points. If $\tilde{u}^{(k)}(y) = g^{(k)}(y)$, then y is called a *$(-)$-boundary straddle point*. A *$(+)$-boundary straddle point* is similarly defined.

DEFINITION 2.7. Let y be a straddle point of dificiency k. The *total deficiency* h of y is defined as k if y is not a cluster point of (e)-points, and as $m + k$ when y is a cluster point of (e)-points and m is defined by Definition 2.6.

Remark. Note that, in view of Lemma 2.5, the total deficiency of a straddle point y is independent of the choice of $\tilde{u}$.

We return now to the characterization problem, and recall that unicity has been established for the case where there are n straddle points. Hence, we may assume in the subsequent discussion that the number of straddle points is smaller than n.

Let $u \in Z(\mathcal{F}; f, g)$ and let E_u be the set of its (e)-points.

We define now a mapping $x(t)$ from E_u into the set of finite subsets of the real line with the following properties:

(1) $x(t)$ is monotone, i.e., if $t_1 < t_2$ then $\max x(t_1) < \min x(t_2)$.

(2) If t is an (e)-point which is not a straddle point, then $x(t)$ is a single point. If t is a non-boundary straddle point of total deficiency h_t, then $x(t)$ consists of h_t points. If t is a boundary straddle point of total deficiency h_t, then $x(t)$ consists of $h_t + 1$ points.

An explicit formula for a mapping with these properties is given by

$$\begin{aligned} x(t) &= t + H_t + l_t, && \text{if } t \text{ is not a straddle point,} \\ &= \bigcup_{j=0}^{h_t - 1} \{t + H_t + l_t + j\}, && \text{if } t \text{ is a non-boundary straddle point,} \\ &= \bigcup_{j=0}^{h_t} \{t + H_t + l_t + j\}, && \text{if } t \text{ is a boundary straddle point.} \end{aligned} \tag{2.10}$$

Here h_t is the total deficiency of t, $H_t = \sum_{s<t} h_s$, and l_t is the number of boundary straddle points that are smaller than t.

We consider the range of $x(E_u)$ as a subset of $\mathbb{R}$ and define a function α on this set as follows: $\alpha(s)$ has the value $+1$ (-1, resp.) if the point s is in one of the following categories: (a) s is the image of a $(+)$-point ($(-)$-point, resp.), (b) s is the rightmost point of $x(t)$, where t is a $(+)$-boundary ($(-)$-boundary, resp.) straddle point, (c) s is the leftmost point of $x(1)$, if 1 is a $(+)$-boundary ($(-)$-boundary, resp.) straddle point; $\alpha(s)$ has the value 0 otherwise.

We recall now some notation concerning sign changes of real valued sequences and functions (cf. [12], where the concept is extensively utilized).

Notation. 1. Let $\bar{x} = (x_1, \ldots, x_N)$ be a finite sequence of real numbers. Then $S^+(\bar{x})$ denotes the maximal number of sign changes of the sequence where the zeros (if they appear) are assigned arbitrary signs.

For example, $S^+[(1, 0, 0, 1)] = 2$, $S^+[(1, 0, 0, 0)] = 3$.

2. Let α be a real function defined on a subset A of the real line. Then

$$S^+(\alpha) = \sup\{S^+[\alpha(t_1), \ldots, \alpha(t_N)]\},$$

where the supremum is taken over all N and over all choices of ordered N-tuples from A.

We are now ready to fully characterize the case of uniqueness.

THEOREM 2.8. *Let $f, g \in C^{(n)}(I)$, $f \geqslant g$, and let $\mathcal{F}$ be an extended n-unisolvent family. Then the set $Z(\mathcal{F}; f, g)$ is a singleton if and only if either*

$$\text{(a)} \quad \sum_{i=1}^{r} h_i \geqslant n, \tag{2.11}$$

where $h_1, \ldots, h_r$ are the total deficiencies of the straddle points, or

(b) *there exists a function* $u^* \in Z(\mathcal{F}; f, g)$ *such that*

$$S^+(\alpha) \geqslant n, \tag{2.12}$$

where α *is the function corresponding to* u^*.

Remark 1. Note that the theorem implies that when there are no straddle points, the function u^* is the only element of $Z(\mathcal{F}; f, g)$ if and only if there exists an $(n+1)$-alternance.

Remark 2. The proof carries over, mutatis mutandis, to the case where f, g are non-smooth. The technical modifications involve the use of one-sided Dini derivatives.

Proof. Sufficiency: Assume first that (2.11) holds. Then we have n Hermite-type conditions that u^* must satisfy in order to be in $Z(\mathcal{F}; f, g)$. Since $\mathcal{F}$ is an extended n-unisolvent system, we conclude that these conditions determine u^* uniquely.

Assume next that there exists a function $u^* \in Z(\mathcal{F}; f, g)$ such that $S^+(\alpha) \geqslant n$. Let $z_1, \ldots, z_{n+1}$ be a sequence of points of $x(E_{u^*})$ for which $S^+[(\alpha(z_1), \ldots, \alpha(z_{n+1}))] = n$.

Let u be any other function in $Z(\mathcal{F}; f, g)$, and consider the difference $v = u - u^*$. We will prove that $v \equiv 0$. Observe that although v is not a function of $\mathcal{F}$, it has to vanish identically if it has n zeros (counting multiplicities). Indeed, if v has n zeros then u^* and u satisfy the same n Hermite data, and therefore must coincide since they belong to an extended n-unisolvent family.

Consider the ordered sequence $z_1, \ldots, z_{n+1}$. If z_i is the image of a $(+)$-point, then $v[x^{-1}(z_i)] \geqslant 0$. Similarly, if z_i is the image of a $(-)$-point, then $v[x^{-1}(z_i)] \leqslant 0$. Note that if $\tilde{t}$ is a non-boundary straddle point with total deficiency h then there are at most h points in the z_i-sequence whose pre-image is $\tilde{t}$, and that

$$v^{(j)}(\tilde{t}) = 0, \qquad j = 0, 1, \ldots, h-1. \tag{2.13}$$

If $\tilde{t}$ is a boundary straddle point, then it has at most $h+1$ image points in the z_i-sequence, and we have $v^{(h)}(\tilde{t}) \geqslant 0$ for a $(+)$-boundary point, $v^{(h)}(\tilde{t}) \leqslant 0$ for a $(-)$-boundary point.

We now construct the vector $(t_1, \ldots, t_{n+1})$ as follows: $\{t_i\}_1^{n+1}$ is a weakly ordered sequence composed of pre-images of the z_i's, according to the following rules: (1) pre-images of the (e)-points which are not straddle points are in $(t_1, \ldots, t_{n+1})$. (2) Let $\tilde{t}$ be a straddle point of total deficiency h, which either is not a boundary straddle point, or is such that $v^{(h)}(\tilde{t}) = 0$. If, in the z_i-sequence, there are j points whose pre-image is $\tilde{t}$, then $\tilde{t}$ will appear in the t_i-sequence j times. (3) Let $\tilde{t}$ be a boundary straddle point of total deficiency h, such that $v^{(h)}(t) \neq 0$. If there are $j \leqslant h$ points whose pre-image is $\tilde{t}$, then $\tilde{t}$ will appear j times in the t_i-sequence.

If, however, there are $h+1$ points in the z_i-sequence whose pre-image is $\tilde{t}$, the point $\tilde{t}$ will appear only h times, and an additional point t' near $\tilde{t}$ will be chosen. If $\tilde{t}=1$, then $t'<\tilde{t}$, whereas if $\tilde{t}\neq 1$, $\tilde{t}<t'$. We observe that if t' is sufficiently near $\tilde{t}$, the sign of $v(t')$ is positive if $\tilde{t}$ is a $(+)$-boundary point, and is negative if $\tilde{t}$ is a $(-)$-boundary point.

The conformity of signs between the $v(t_i)$'s and the $\alpha(z_i)$'s implies now that

$$S^+[v(t_1),\ldots, v(t_{n+1})]=n. \tag{2.14}$$

Let $v(t_p)$ be the first non-zero entry in this sequence. If such an entry does not exist, then $v(t)$ has more than n zeros (counting multiplicities), and the proof is complete. Thus, v has $p-1$ zeros, $\{t_i\}_1^{p-1}$, in $[t_1, t_p)$. Next let $v(t_q)$, $q\geqslant p+1$, be the last entry in the chain of non-zero entries following $v(t_p)$. By (2.14), the values $v(t_p),\ldots, v(t_q)$ alternate in sign, so that continuity implies the existence of $q-p$ zeros in (t_p, t_q). We have therefore $q-1$ zeros in $[t_1, t_q)$. If $q=n+1$, the proof is finished. If not, $v(t_{q+1})=0$, and we have to examine two possibilities:

(i) $v(t_i)=0$, $i\geqslant q+1$. In this case we are assured of $q-1+(n+1)-q=n$ zeros and the proof is finished.

(ii) There exists a first non-zero entry $v(t_r)$, $r>q+1$. It will suffice to show that in (t_q, t_r) there exist $r-q$ zeros, so that in $[t_1, t_r)$ there are $r-1$ zeros. The rest of the proof then follows by repeating (a finite number of times) the steps outlined above.

If $r-q$ is odd then the signs of $v(t_r)$ and $v(t_q)$ are different by (2.14). On the other hand, the number $r-q-1$ of zeros in (t_q, t_r) following from the definition of t_r is even. Thus, there has to be another point of sign change, or a higher multiplicity of one of the zeros. In either case, there will be $r-q$ zeros in (t_q, t_r), concluding the proof.

Necessity: We assume that (2.11) does not hold, and that there exists a function $u_0\in Z(\mathcal{F}; f, g)$ such that $S^+(\alpha)=p<n$. Note that, in view of Theorem 1.7, this implies the existence of straddle points, and $\|f-g\|=2r(\mathcal{F}; f, g)$. We now proceed to exhibit another function u_1, $u_1\neq u_0$, in $Z(\mathcal{F}; f, g)$. The method of proof bears some resemblance to that used in the proof of Theorem 1.7, with appropriate modifications necessitated by the existence of straddle points. We start with the case where no straddle point is a cluster of (e)-points. Let $y_1,\ldots, y_r$, $1\leqslant r$, be the straddle points, and let their deficiencies be $k_1,\ldots, k_r$, with

$$k=\sum_{i=1}^{r} k_i<n. \tag{2.15}$$

Since no y_i is a cluster point, it follows that the total deficiencies in this case are equal to the ordinary deficiencies. For each i, let $\varepsilon(y_i)$ be chosen sufficiently small, so that

$$(y_i - \varepsilon(y_i),\ y_i + \varepsilon(y_i))$$

does not contain any (e)-points except y_i. Let

$$\varepsilon = \min_i \varepsilon(y_i), \quad v_i = (y_i - \varepsilon,\ y_i + \varepsilon), \qquad i = 1,\ldots, n.$$

Let $I_0 = [0, 1] \backslash \bigcup_{i=1}^{r} v_i$, and observe that, since I_0 is a closed set containing no straddle points, we have

$$2r(u_0; f, g) - \max(|f(t) - g(t)|; t \in I_0) = 5\delta > 0. \tag{2.16}$$

Let (y_i, y_{i+1}), $1 \leqslant i \leqslant r-1$, be an interval between straddle points containing "signed" (e)-points. If $y_i > 0$ or $y_r < 1$ a similar analysis can be carried out for $[0, y_1)$ or $(y_r, 1]$, respectively.

Assume, for concreteness, that (y_1, y_r) contains a $(+)$-point; then it is in I_0, by the construction of the v_i's, and we may assume that the leftmost (e)-point in $(y_1, y_r) \cap I_0$ is a $(+)$-point, which we denote by $t_{i,1}$. Note that

$$t_{i,1} = \min\{t; t \in (y_i, y_{i+1}),\ t \text{ is a } (+)\text{-point}\}.$$

Define

$$\underline{t}_{i,1} = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, f(t) - u_0(t) \geqslant r(u_0; f, g) - 2\delta\}.$$

Consider now two possibilities:

(1) There exist no $(-)$-points in (y_i, y_{i+1}). Then define

$$\bar{t}_{i,1} = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, f(t) - u_0(t) \geqslant r(u_0; f, g) - 2\delta\}.$$

(2) There exist $(-)$-points in (y_i, y_{i+1}). Define

$$t_i' = \min\{t; t \in (y_i, y_{i+1}) \cap I_0,\ t \text{ is a } (-)\text{-point}\},$$

$$\bar{t}_{i,1} = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, t < t_i'; f(t) - u_0(t) \geqslant r(u_0; f, g) - 2\delta\},$$

$$\underline{t}_i' = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, t > \bar{t}_{i,1}; u_0(t) - g(t) \geqslant r(u_0; f, g) - 2\delta\}.$$

Note that by (2.16), $\underline{t}_i' > \bar{t}_{i,1}$. We may now continue this process, depending on the existence of $(+)$-points to the right of t_i'. If there are none, the process is ended by defining

$$\bar{t}_i' = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, u_0(t) - g(t) \geqslant r(u_0; f, g) - 2\delta\}.$$

Otherwise, we define

$$t_{i,2} = \min\{t;\, t \in (y_i, y_{i+1}) \cap I_0,\, t_i' < t,\, t\text{—a (+)-point}\}$$

and continue along the same lines. Note that in view of the finiteness of $S^+(\alpha)$ (we have $S^+(\alpha) < n$, in fact), the process has a finite number of steps.

We apply this procedure for all intervals containing (e)-points. We have thus constructed a set of intervals

$$\{A_j\}_{j=1}^{s}, \qquad \bigcup_{i=1}^{s} A_j \subset I_0,$$

with the following properties:

(a) Each interval contains an (e)-point. All (e)-points are contained in the union of these intervals.

(b) If A_j contains a (+)-point, then

$$f(t) - u_0(t) \geqslant r(u_0; f, g) - 2\delta \qquad \text{for all} \quad t \in A_j. \tag{2.17}$$

We call this A_j a (+)-interval. If A_j contains a (−)-point, then

$$u_0(t) - g(t) \geqslant r(u_0; f, g) - 2\delta \qquad \text{for all} \quad t \in A_j. \tag{2.18}$$

This A_j will be called a (−)-interval.

(c) If $\{A_j, \ldots, A_{j+l}\}$ are in the same interval (y_i, y_{i+1}), then their signs alternate, and there exists an interval of positive length between adjacent A_i's. Choose an ordered sequence in E_{u_0} consisting of one (e)-point from each A_i, and the straddle points. Apply the mapping $x(t)$ to the sequence and construct the vector $\{\alpha(s_i)\}_{i=1}^{N}$. Here $x(t)$ and $\alpha(s)$ are as defined prior to Theorem 2.8. Note that $S^+[(\alpha(s_i), \ldots, \alpha(s_N))] = p < n$.

We will show that there exists a function u_1, $u_1 \neq u_0$, in $Z(\mathcal{F}; f, g)$. We start by noting that u_1 has to satisfy the p conditions implied by the fact that $y_1, \ldots, y_r$ are straddle points, with corresponding multiplicities $k_1, \ldots, k_r$, viz.,

$$u_1^{(j)}(y_i) = u_0^{(j)}(y_i), \qquad i = 1, \ldots, r;\, j = 0, \ldots, k_i - 1. \tag{2.19}$$

Consider next a sequence of consecutive zeros in $\{\alpha(s_i)\}_{i=1}^{N}$. Suppose there are l zeros. These may correspond to the deficiency of one straddle point, or to the combined deficiencies of several consecutive straddle points, where no intervening (+)- or (−)-points exist. There are two possibilities: (1) The l zeros are an initial or a final segment of the vector $\{\alpha(s_i)\}_{i=1}^{N}$. In this case we do not impose additional conditions on u at the corresponding straddle points. (2) On both sides of the segment of zeros, there exist nonzero terms. Let the adjacent sign from the left (right) be denoted by $(\text{sgn})_{\text{L}}$ $[(\text{sgn})_{\text{R}}$,

respectively]. If $(-1)^l(\text{sgn})_L(\text{sgn})_R = 1$, then no additional requirements are imposed on u_1 at the corresponding straddle points. If, however, $(-1)^l(\text{sgn})_L(\text{sgn})_R = -1$, then we require

$$u_1^{(k_{i^*})}(y_{i^*}) = u_0^{(k_{i^*})}(y_{i^*}), \tag{2.20}$$

where y_{i^*} is the first straddle point corresponding to the block of l zeros.

Consider finally two adjacent non-zero terms. If the signs are identical (this may happen only if at least one of the signs stems from a "signed"-boundary straddle point) then no additional requirements are imposed on u_1. Suppose the terms are of opposite signs. This can happen when both correspond to (e)-points chosen from adjacent A_i's, say A_m, A_{m+1}, or when at least one of the points is a "signed"-boundary straddle point. In the first case, we choose a point t^* in $(\max A_m, \min A_{m+1})$ and require

$$u_1(t^*) = u_0(t^*). \tag{2.21}$$

In the second case, assume that the first of the two terms corresponds to a straddle point $\tilde{y}$. We then require

$$u_1(\tilde{y} + \tfrac{3}{4}\varepsilon) = u_0(\tilde{y} + \tfrac{3}{4}\varepsilon). \tag{2.22}$$

Observe that the total number of zeros prescribed for $u_1 - u_0$ by the conditions of the form (2.19)–(2.22) is equal to $S^+(\alpha) = p$. Indeed, consider the case where $(-1)^l(\text{sgn})_L(\text{sgn})_R = -1$. The contribution of the sequence of l zeros to $S^+(\alpha)$ is then $l+1$, and we have, in (2.20), adjoined one zero to the l zeros prescribed by (2.19). The other cases are even simpler.

We now impose $n - p - 1$ additional conditions of coincidence at 0,

$$u_1^{(\mu+j)}(0) = u_0^{(\mu+j)}(0), \qquad j = 0, 1, \ldots, n-p-2, \tag{2.23}$$

where μ is the smallest derivative at 0 not previously prescribed.

Finally, if there exist "signed" (e)-points or "signed"-boundary straddle points, then we choose one such point $\tilde{\tilde{t}}$, and impose an n-th condition of the form

$$u_1^{(\nu)}(\tilde{\tilde{t}}) = u_0^{(\nu)}(\tilde{\tilde{t}}) + \eta, \tag{2.24}$$

where ν is the smallest derivative at $\tilde{\tilde{t}}$ not previously prescribed, and η is a small number whose sign agrees with the "sign" of the point. If there exist no "signed" points, we choose any straddle point $\tilde{\tilde{t}}$ and require (2.24) with $\eta > 0$.

Since $\mathcal{F}$ is an extended u-unisolvent family, there exists a (unique) u_1 satisfying all of the n above-mentioned conditions. Furthermore, $u_1 \neq u$ by (2.24), so that $u_1 - u$ can have no additional zeros (counting multiplicities) besides the $n-1$ zeros prescribed in the construction.

Hence, $u_1 - u$ changes sign exactly at the interior zeros of odd multiplicity. It follows that $u_1 > u_0$ on each $(+)$ interval, $u_1 < u_0$ on each $(-)$-interval. Furthermore, if y is a $(+)$-boundary straddle point of deficiency k, then $u_1 > u_0$ in the vicinity of y, so that in view of (2.19), we must have $u_1^{(k)}(y) > u_0^{(k)}(y)$. The case of $(-)$-boundary straddle points is similarly handled. Finally, if η is chosen to be sufficiently small, then by the continuity property of elements of $\mathscr{F}$ (Lemma 1.2) we have $\|u_0 - u_1\| < \delta$, so that

$$\max\left[|u_1(t) - f(t)|, |u_1(t) - g(t)|;\ t \in [0,1]\Big\backslash\left(\bigcup_{i=1}^{r} v_i\right)\Big\backslash\left(\bigcup_{i=1}^{r} A_i\right)\right] < r(u_0; f, g).$$

Collecting these results, we deduce that $u_1 \in Z(\mathscr{F}; f, g)$, completing the proof in the case where no straddle point is a cluster point of (e)-points.

We consider now the general case, and describe the necessary adjustments in the proof. Let y be a straddle point which is a cluster point of (e)-points. As we have noted before, the finiteness of $S^+(\alpha)$ implies that if $\varepsilon_1 > 0$ is sufficiently small, then in $(\tilde{y} - \varepsilon_1, \tilde{y})$ all (e)-points are of one sign, and in $(\tilde{y}, \tilde{y} + \varepsilon_1)$ all (e)-points are of one sign (not necessarily the same sign as before). Note in passing that if $\tilde{y}$ is a $(-)$-boundary point, then it can be a cluster point of $(-)$-points only, by the analysis in the proof of Lemma 2.5. The analogous result holds for $(+)$-boundary points.

Choose ε_1 as above, and let $\tilde{z}$, $\tilde{w}$ be the largest (e)-point in $(\tilde{y}, \tilde{y} + \varepsilon_1)$ and the smallest (e)-point in $(\tilde{y} - \varepsilon_1, \tilde{y})$, respectively. Let $\varepsilon(\tilde{y}) = \min\{(\tilde{z} - \tilde{y}, \tilde{y} - \tilde{w})\}$, and let

$$\tilde{\varepsilon} = \min\{\varepsilon(\tilde{y});\ \tilde{y} \text{ is a cluster point of } (e)\text{-points}\}$$

$$\varepsilon^* = \min\{\varepsilon(y_i);\ y_i \text{ is not a cluster point of } (e)\text{-points}\},$$

$$\varepsilon = \min(\tilde{\varepsilon}, \varepsilon^*),$$

where $\varepsilon(y_i)$ is as defined in the beginning of the proof of the necessity part. Define next v_i, I_0 as before and the rest of the proof can be carried out with no further modifications. Q.E.D.

We have shown that, in contrast to the situation where one function is approximated, the Chebyshev center of a set is not necessarily a singleton. We will now record some simple observations concerning the set of pairs for which $Z(\mathscr{F}; f, g)$ is a singleton.

We consider the space of pairs of functions (f, g), $f, g \in C[0,1]$, and define $\rho[(f, g), (\tilde{f}, \tilde{g})] = \max[\|f - \tilde{f}\|, \|g - \tilde{g}\|]$.

Assertion 2.9. Let f, g, $f \geqslant g$, be a pair such that $Z(\mathscr{F}; f, g)$ is not a singleton. Then, for each $\varepsilon > 0$, there exists another $(\tilde{f}, \tilde{g})$ such that $\rho[(f, g), (\tilde{f}, \tilde{g})] < \varepsilon$, and $Z(\mathscr{F}; f, g)$ is not a singleton.

Proof. Let $\bar{u} \in Z(\mathcal{F}; f, g)$. There exist r straddle points, $y_1, \ldots, y_r$, with total deficiencies $h_1, \ldots, h_r$, $l = \sum_1^r h_i < n$. Perturb f slightly downward on one interval not containing straddle points, obtaining $\tilde{f}$ in this way. Then clearly $\bar{u} \in Z(\mathcal{F}; \tilde{f}, g)$, the r straddle points remain the only straddle points, and no new (e)-points are created. Hence $Z(\mathcal{F}; \tilde{f}, g)$ is not a singleton. Q.E.D.

Remark. The same proof shows that if $Z(\mathcal{F}; f, g)$ is a singleton, but there exist straddle points, then there exists a pair $(\tilde{f}, \tilde{g})$ near (f, g) for which $Z(\mathcal{F}; \tilde{f}, \tilde{g})$ is not a singleton.

However, the situation is different if $(\mathcal{F}; f, g)$ has no straddle points. The following assertion can be easily established, using straightforward continuity arguments.

Assertion 2.10. Let $Z(\mathcal{F}; f, g)$ be a singleton, and assume no straddle points exist. Then there exists a neighborhood V of (f, g), such that for each pair $(\tilde{f}, \tilde{g})$ in V, the center $Z(\mathcal{F}; \tilde{f}, \tilde{g})$ is a singleton, and no straddle points exist.

Using the standard methods, we can deduce a local continuity property for the "best approximation" operator defined for such pairs, viz.,

Assertion 2.11. Let (f, g) be a pair such that $(\mathcal{F}; f, g)$ has no straddle points. Let T be defined on the set of such pairs by $T(f, g) = Z(\mathcal{F}; f, g)$. Then T is continuous at (f, g).

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